

JOURNAL OF ALGEBRA **140**, 502–518 (1991)Residual Division, Semi-Prime Operations,
and a Theorem of Sakuma and Okuyama

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Received September 15, 1989

Let I be a regular ideal of a Noetherian ring R . Then it is well known that: (a) $I^{n+k} : I^n = I^k$ for all large k and for all $n \geq 0$; (b) if I is principal and H is another ideal of R , then $I^{n+j}H^m : I^j = I^nH^m = I^n(I^jH^m : I^j)$ for all $m \geq 0$, $j \geq 0$, and $n \geq 1$; and (c) if R is local and analytically unramified, then $(I^{n+k})_a = (I^k)_a$ for all large k and for all $n \geq 0$, where $(I^j)_a$ is the integral closure of I^j . The main results in this paper generalize these three theorems to the case where H and I are finite collections of Noetherian filtrations on R , and these new results are then used to show that a semi-local ring R is analytically unramified if and only if for every regular ideal I of R there exists a regular ideal K of R such that $(I^n)_a = I^nK : K$ for all $n \geq 1$. © 1991 Academic Press, Inc.

1. INTRODUCTION

All rings in this paper are assumed to be commutative with identity (and usually also Noetherian), and the terminology is generally standard.

Theorems (a) and (c) of the abstract have been useful in a number of research problems in Commutative Algebra. The first of these has been used to derive some new results concerning asymptotic prime divisors, and (c) (the theorem of Sakuma and Okuyama) has been useful in problems concerning analytically unramified semi-local rings. Also, (a) and (c) are related, since $I_a = \cup \{IK : K; K \text{ is a regular ideal of } R\}$ (and since $I^{n+k} : I^n = \cup \{I^{i+k} : I^i; i = 1, \dots, n\}$). Further, even though (b) is obvious, if one does not assume that I is principal, then it clearly generalizes (a), so all three results can be viewed as being specific cases of the semi-prime operation $I \rightarrow I_{\mathcal{A}} = \cup \{IK : K; K \in \mathcal{A}\}$, where \mathcal{A} is a multiplicatively closed set of nonzero ideals of R , and they show that by using large powers of I the Artin-Rees phenomenon occurs; that is, there exists a positive integer k such that I^{n-k} factors out for $n \geq k$. This observation was the starting point for this paper, and since several recent papers have extended results for ideals to finite collections of ideals and/or Noetherian filtrations, it

seemed natural to try such an extension on these three results and also to see if some \mathcal{A} -version of (c) holds for all Noetherian rings. Our main results show that such extensions do hold.

To be more specific, let $\Phi = (\phi_1, \dots, \phi_f)$ (resp., $\mathbf{I} = (I_1, \dots, I_g)$) be a finite collection of Noetherian filtrations (resp. ideals) of a Noetherian ring R . Then, concerning (b) of the abstract, in (2.3.2) it is shown that if each I_i is regular, then $\mathbf{I}^{n+j}\Phi(\mathbf{m}) : \mathbf{I}^j = \mathbf{I}^n\Phi(\mathbf{m})$ for all large $n \in \mathbf{P}_g$ and for all $j \in \mathbf{N}_g$ and $\mathbf{m} \in \mathbf{N}_f$ (see (2.2) for the definitions and notation), and the special case where $\mathbf{m} = 0 \in \mathbf{N}_f$ extends (a) of the abstract (see (2.3.3)). Then in (2.4.2) and (2.4.3) these results are extended to the case where the I_i need not be regular. In Section 3 the theorem of Sakuma and Okuyama ((c) of the abstract) is generalized to if R is locally analytically unramified and has finite integral closure and if $\phi \rightarrow \phi_x$ is a semi-prime operation on the set of filtrations ϕ on R such that $\phi_x \leq \phi_w$ (see (3.2) for the definitions), then $(\mathbf{I}^{n+k}\Phi(\mathbf{m}))_x = \mathbf{I}^n(\mathbf{I}^k\Phi(\mathbf{m}))_x$ for large $k \in \mathbf{P}_g$ and for all $n \in \mathbf{N}_g$ and $\mathbf{m} \in \mathbf{N}_f$. In Section 4 this same conclusion is shown to hold for all Noetherian rings when it is assumed only that $\phi_x \leq \phi_{\mathcal{A}}$ for some finitely generated multiplicatively closed set \mathcal{A} of regular ideals of R . Finally, these results are used in Section 5 to show that a semi-local ring R is analytically unramified if and only if for all finite collections $\Phi = (\phi_1, \dots, \phi_f)$ of regular Noetherian filtrations on R there exists a regular ideal K of R such that $(\Phi(\mathbf{n}))_{\mathcal{A}} = \Phi(\mathbf{n})K : K$ for all $\mathbf{n} \in \mathbf{N}_f$.

2. A PROPERTY OF RESIDUAL DIVISION

If I is a regular ideal of a Noetherian ring R , then $I^{n+k} : I^n = I^k$ for all large k and $n \geq 0$, and if I is principal and H is another ideal of R , then $H^m I^{n+j} : I^j = H^m I^n$ for all $m \geq 0$, $j \geq 0$, and $n \geq 1$. The main result in this section, (2.3), generalizes these two results to finite collections of Noetherian filtrations on R , and its corollary (2.5) shows that the second of these holds for all regular ideals I when n is large. To prove (2.3) we need the following lemma and definitions.

(2.1) LEMMA. *Let H , I , and J be ideals in a Noetherian ring R such that I is regular and $\text{Rad}(I) \subseteq \text{Rad}(J)$. Then:*

(2.1.1) $H I^n : J \subseteq H$ for all large n .

(2.1.2) If $H = bR$ is a regular principal ideal, then $H^m I^n : J = H^m (I^n : J)$ for all $m \geq 0$ and for all large n .

(2.1.3) If b_1, \dots, b_g are regular nonunits in R , then $b_1^{m_1} \dots b_g^{m_g} I^n : J = b_1^{m_1} \dots b_g^{m_g} (I^n : J)$ for all nonnegative integers m_1, \dots, m_g and for all large n .

(2.1.4) $I^{n+1} : I = I^n$ for all large n .

Proof. For (2.1.1) let $\bigcap_{i=1}^m Q_i$ be a normal primary decomposition of H , let $P_i = \text{Rad}(Q_i)$ for $i = 1, \dots, m$, and let L_i be the completion of R_{P_i} , so $Q_i L_i$ is a $P_i L_i$ -primary component of HL_i . For each $i = 1, \dots, m$ we now consider the two cases: (a) $I \not\subseteq P_i$ and (b) $I \subseteq P_i$. If (a) holds, then $\text{Rad}(I) \subseteq \text{Rad}(J)$ implies that $J \not\subseteq P_i$, so $HI^n L_i : JL_i = HL_i \subseteq Q_i L_i$ for all $n \geq 1$. And, if (b) holds, then $\{HI^n L_i : JL_i; n \geq 1\}$ is a decreasing sequence of ideals of L_i (possibly, $JL_i = L_i$), and $\bigcap \{HI^n L_i : JL_i; n \geq 1\} = (\bigcap \{HI^n L_i; n \geq 1\}) : JL_i = (0) : JL_i$, and $(0) : JL_i = (0)$ since JL_i is regular (since $\text{Rad}(I) \subseteq \text{Rad}(J)$ and I is regular it is implied that J is regular). Therefore, since $P_i^k L_i \subseteq Q_i L_i$ for some positive integer k , and since L_i is complete, it follows from [3, (30.1)] that $HI^n L_i : JL_i \subseteq P_i^k L_i \subseteq Q_i L_i$ for all large n . Thus, it follows that $HI^n : J \subseteq \bigcap \{(HI^n L_i : JL_i) \cap R; i = 1, \dots, m\} \subseteq \bigcap \{Q_i L_i \cap R; i = 1, \dots, m\} = \bigcap_{i=1}^m Q_i = H$ for all large n , and hence (2.1.1) holds.

For (2.1.2), it follows from (2.1.1) that $bI^n : J \subseteq bR$ for all large n . By induction on $m \geq 1$ assume that $m > 1$ and that $b^{m-1}I^n : J \subseteq b^{m-1}R$. Then $b^m I^n : J \subseteq b^{m-1}R$, so $b^m I^n : J = (b^m I^n : J) \cap b^{m-1}R = b^{m-1}[(b^m I^n : J) : b^{m-1}R] = b^{m-1}(b^m I^n : b^{m-1}J) = b^{m-1}(bI^n : J) \subseteq b^{m-1}bR = b^m R$. Therefore $b^m I^n : J \subseteq b^m R$ for all $m \geq 1$ and for all large n , so $b^m I^n : J = (b^m I^n : J) \cap b^m R = b^m(b^m I^n : b^m J) = b^m(I^n : J)$ for all $m \geq 1$ and for all large n , so (2.1.2) holds.

For (2.1.3), (2.1.2) shows that for $i = 1, \dots, g$, $b_i^{m_i} I^n : J = b_i^{m_i}(I^n : J)$ for all $m_i \geq 0$ and for all large n . Therefore if n is large, then $b_1^{m_1} \dots b_g^{m_g} I^n : J \subseteq b_1^{m_1} R$, so $b_1^{m_1} \dots b_g^{m_g} I^n : J = (b_1^{m_1} \dots b_g^{m_g} I^n : J) \cap b_1^{m_1} R = b_1^{m_1}(b_1^{m_1} \dots b_g^{m_g} I^n : b_1^{m_1} J) = b_1^{m_1}(b_2^{m_2} \dots b_g^{m_g} I^n : J)$. So by repeating this with $b_2^{m_2} \dots b_g^{m_g} I^n : J$ in place of $b_1^{m_1} \dots b_g^{m_g} I^n : J$, etc., the conclusion follows after $g - 1$ repetitions, so (2.1.3) holds.

Finally, since I is regular, there exists a regular superficial element of degree h (for some positive integer h) for I , so there exists a positive integer c such that $(I^{n+h} : bR) \cap I^c = I^n$ for all large n . Also, by (2.1.1) (with $H = I^c$ and $J = I^h$), $I^c I^{n+h-c} : I^h \subseteq I^c$ for all large $n + h - c$; that is, $I^{n-h} : I^h \subseteq I^c$ for all large n . Therefore it follows that $I^n \subseteq I^{n+1} : I \subseteq \dots \subseteq I^{n+h} : I^h \subseteq (I^{n+h} : bR) \cap I^c = I^n$, and hence $I^{n+1} : I = I^n$ for all large n , so (2.1.4) holds. Q.E.D.

(2.2) DEFINITION. If R is a ring, then:

(2.2.1) A *filtration* $\phi = \{\phi(n)\}_{n \geq 0}$ on R is a descending sequence of ideals $\phi(n)$ of R such that $\phi(0) = R$ and $\phi(i)\phi(j) \subseteq \phi(i+j)$ for all non-negative integers i and j .

(2.2.2) The *product* $\phi_1 \dots \phi_g$ of g filtrations ϕ_1, \dots, ϕ_g on R is the sequence of ideals $\phi_1 \dots \phi_g = \{\phi_1(n) \dots \phi_g(n)\}_{n \geq 0}$. (It is readily checked that $\phi_1 \dots \phi_g$ is a filtration on R .)

(2.2.3) If ϕ_1, \dots, ϕ_g are filtrations on R , then the *Rees ring* $\mathbf{R}(R, \phi_1, \dots, \phi_g)$ of R with respect to ϕ_1, \dots, ϕ_g is the graded subring $\mathbf{R}(R, \phi_1, \dots, \phi_g) = R[u_1, \dots, u_g, \{t_1^i \phi_1(i)\}_{i=1}^\infty, \dots, \{t_g^i \phi_g(i)\}_{i=1}^\infty]$ of $R[u_1, \dots, u_g, t_1, \dots, t_g]$, where t_1, \dots, t_g are algebraically independent over R and $u_i = 1/t_i$ for $i = 1, \dots, g$. If each ϕ_i is the sequence of powers of an ideal I_i , then we use $\mathbf{R}(R, I_1, \dots, I_g)$ in place of $\mathbf{R}(R, \phi_1, \dots, \phi_g)$.

(2.2.4) A filtration ϕ on R is said to be *Noetherian* in case $\mathbf{R}(R, \phi)$ is a Noetherian ring (see (2.2.3)).

(2.2.5) \mathbf{P}_g (resp. $\mathbf{N}_g, \mathbf{Z}_g$) is the set of all g -tuples of positive (resp. nonnegative, all) integers. If $\mathbf{n} = (n_1, \dots, n_g) \in \mathbf{Z}_g$, then $\mathbf{n}(i)$ denotes n_i , the i th component of \mathbf{n} , and it is said that \mathbf{n} is *large* in case each $\mathbf{n}(i)$ is large. Also, if \mathbf{m} and \mathbf{n} are in \mathbf{N}_g and h is a positive integer, then $\mathbf{m} + \mathbf{n}$, $\mathbf{m} - \mathbf{n}$, $\mathbf{m}\mathbf{n}$, and $h\mathbf{n}$ are defined in the usual componentwise manner, but we use only $\mathbf{m} - \mathbf{n}$ when $\mathbf{m} \geq \mathbf{n}$ (that is, $\mathbf{m}(i) \geq \mathbf{n}(i)$ for $i = 1, \dots, g$). Further, $\mathbf{0}$ (resp. $\mathbf{1}$) denotes the element $(0, 0, \dots, 0) \in \mathbf{N}_g$ (resp. $(1, 1, \dots, 1) \in \mathbf{P}_g$), and \mathbf{e}_i is the element in \mathbf{N}_g such that $\mathbf{e}_i(i) = 1$ and $\mathbf{e}_i(j) = 0$ for $j \neq i$.

(2.2.6) If $\mathbf{b} = (b_1, \dots, b_g)$ resp. $\mathbf{I} = (I_1, \dots, I_g)$, $\Phi = (\phi_1, \dots, \phi_g)$ is a collection of g elements (resp. ideals, filtrations) of a ring related to R and $\mathbf{m} \in \mathbf{N}_g$, then $\mathbf{b}^{\mathbf{m}}$ denotes the element $b_1^{\mathbf{m}(1)} \dots b_g^{\mathbf{m}(g)}$, and $\mathbf{I}^{\mathbf{m}}$ (resp., $\Phi(\mathbf{m})$) denotes the ideal $I_1^{\mathbf{m}(1)} \dots I_g^{\mathbf{m}(g)}$ (resp. $\phi_1(\mathbf{m}(1)) \dots \phi_g(\mathbf{m}(g))$).

Concerning (2.2.1), it should be noted that filtrations are a very useful generalization of the sets of powers of an ideal I in a ring R , and there are many important filtrations that are generally not such powers of an ideal. (For example, $\{Q^{(n)}\}_{n \geq 0}$, where $Q^{(n)}$ is the n th symbolic power of the primary ideal Q ; $\{(I^n)_a\}_{n \geq 0}$, where $(I^n)_a$ is the integral closure in R of I^n ; and $\{u^n \mathbf{A} \cap R\}_{n \geq 0}$, where \mathbf{A} is a graded subring of $R[u, t]$ that contains $R[u, tI]$ for a given ideal I of R .)

Theorem (2.3) generalizes one of the main results in [2], where it is shown that the conclusions hold with $J^{\mathbf{m}}$ in place of $\Phi(\mathbf{m})$ (where J is an ideal of R), and we need this strengthened version for the proof of (4.4). Instead of trying to describe how to amend the proof in [2] to get this strengthened version, it is easier to simply prove it anew. (And our new proof of (2.3) is quite different from the proof of the corresponding result in [2].)

(2.3) THEOREM. If $\Phi = (\phi_1, \dots, \phi_f)$ is a finite collection of Noetherian filtrations on a Noetherian ring R and $\mathbf{I} = (I_1, \dots, I_g)$ is a finite collection of ideals of R , then:

(2.3.1) If I_g is regular, then for all large integers k it holds that $\mathbf{I}^{n+j\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^j = \mathbf{I}^n\Phi(\mathbf{m})$ (see (2.2.5) and (2.2.6) for all $\mathbf{n} \in \mathbf{N}_g$ such that $\mathbf{n}(g) \geq k$, for all integers $j \geq 0$, and for all $\mathbf{m} \in \mathbf{N}_f$.

(2.3.2) If I_1, \dots, I_g are regular, then for all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $\mathbf{I}^{n+\mathbf{j}}\Phi(\mathbf{m}) : \mathbf{I}^{\mathbf{j}} = \mathbf{I}^n\Phi(\mathbf{m})$ for all $\mathbf{n} \geq \mathbf{k}$, for all $\mathbf{j} \in \mathbf{N}_g$, and for all $\mathbf{m} \in \mathbf{N}_f$.

(2.3.3) If I_1, \dots, I_g are regular, then for all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $\mathbf{I}^{n+\mathbf{j}} : \mathbf{I}^{\mathbf{j}} = \mathbf{I}^n$ for all $\mathbf{n} \geq \mathbf{k}$ and for all $\mathbf{j} \in \mathbf{N}_g$.

Proof. Assume that (2.3.1) holds and that each I_i is regular. Then by applying (2.3.1) g times (the i th time with $I_{g-(i+1)}$ in place of I_g) it follows that the following holds for $i = 1, \dots, g$ and for all large integers k_i

$$\mathbf{I}^{n+j\mathbf{e}_i}\Phi(\mathbf{m}) : I_i^j = \mathbf{I}^n\Phi(\mathbf{m}) \text{ for all } \mathbf{n} \in \mathbf{N}_g \text{ such that } \mathbf{n}(i) \geq k_i, \text{ for all integers } j \geq 0, \text{ and for all } \mathbf{m} \in \mathbf{N}_f. \quad (\#)$$

Therefore fix large integers k_1, \dots, k_g , let $\mathbf{k} = (k_1, \dots, k_g)$, let $\mathbf{n} \geq \mathbf{k}$, let $\mathbf{j} \in \mathbf{N}_g$, and let $\mathbf{m} \in \mathbf{N}_f$. Then $\mathbf{I}^{n+\mathbf{j}}\Phi(\mathbf{m}) : \mathbf{I}^{\mathbf{j}} = (\mathbf{I}^{n+\mathbf{j}}\Phi(\mathbf{m}) : I_1^{j(1)}) : \mathbf{I}^{1-j(1)\mathbf{e}_1} = \mathbf{I}^{n+\mathbf{j}-j(1)\mathbf{e}_1}\Phi(\mathbf{m}) : \mathbf{I}^{j-j(1)\mathbf{e}_1}$, by $(\#)$. It follows from $g-1$ more applications of $(\#)$ that (2.3.2) holds, so (2.3.2) follows from (2.3.1), and (2.3.3) is the special case $\mathbf{m} = 0 \in \mathbf{N}_f$ of (2.3.2), so it suffices to prove (2.3.1).

For this, let $\mathbf{R} = \mathbf{R}(R, \phi_1, \dots, \phi_{f+g-1})$, where $\phi_{f+i} = \{(I_i)^n\}_{n \geq 0}$ for $i = 1, \dots, g-1$. Also, let $I = I_g$, so I is regular. Therefore if $\mathbf{u} = (u_1, \dots, u_{f+g-1})$, then (2.1.3) implies that $\mathbf{u}^m I^n \mathbf{R} : I \mathbf{R} = \mathbf{u}^m (I^n \mathbf{R} : I \mathbf{R})$ for all $\mathbf{m} \in \mathbf{N}_{f+g-1}$ and for all large n . Also, (2.1.4) shows that $I^{n+1} \mathbf{R} : I \mathbf{R} = I^n \mathbf{R}$ for all large n . It follows that for all large integers k it holds that $\mathbf{u}^m I^{n+1} \mathbf{R} : I \mathbf{R} = \mathbf{u}^m I^n \mathbf{R}$ for all $\mathbf{m} \in \mathbf{N}_{f+g-1}$ and for all $n \geq k$, so by contracting these ideals to R it follows that $I^{n+1} \Gamma(\mathbf{m}) : I = I^n \Gamma(\mathbf{m})$ for all $\mathbf{m} \in \mathbf{N}_{f+g-1}$ and for all $n \geq k$, where $\Gamma = (\phi_1, \dots, \phi_{f+g-1})$, and (2.3.1) readily follows from this. Q.E.D.

Corollary (2.4) extends (2.3) to the case when the I_i are not regular.

(2.4) COROLLARY. Let $\Phi = (\phi_1, \dots, \phi_f)$, $\mathbf{I} = (I_1, \dots, I_g)$, and R be as in (2.3). Then:

(2.4.1) If $Z = \cup \{(0) : I_g^j : j \geq 1\} \neq R$, then for all large integers k it holds that $\mathbf{I}^{n+j\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^j = \mathbf{I}^n\Phi(\mathbf{m}) + Z$ for all $\mathbf{n} \in \mathbf{N}_g$ such that $\mathbf{n}(g) \geq k$, for all large integers j , and for all $\mathbf{m} \in \mathbf{N}_f$.

(2.4.2) If $Z = \cup \{(0) : \mathbf{I}^{\mathbf{j}} : \mathbf{j} \in \mathbf{N}_g\} \neq R$, then for all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $\mathbf{I}^{n+\mathbf{j}}\Phi(\mathbf{m}) : \mathbf{I}^{\mathbf{j}} = \mathbf{I}^n\Phi(\mathbf{m}) + Z$ for all $\mathbf{n} \geq \mathbf{k}$, for all large $\mathbf{j} \in \mathbf{P}_g$, and for all $\mathbf{m} \in \mathbf{N}_f$.

(2.4.3) If $Z = \cup \{(0) : \mathbf{I}^{\mathbf{j}} : \mathbf{j} \in \mathbf{N}_g\} \neq R$, then for all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $\mathbf{I}^{n+\mathbf{j}} : \mathbf{I}^{\mathbf{j}} = \mathbf{I}^n + Z$ for all $\mathbf{n} \geq \mathbf{k}$ and for all large $\mathbf{j} \in \mathbf{P}_g$.

Proof. For (2.4.1) the ideal I_g/Z is regular in R/Z , so (2.3.1) shows that for all large integers k it holds that

$$(\mathbf{I}^1/Z)^{\mathbf{n}+j\mathbf{e}_g}(\Phi(\mathbf{m})/Z) : (I_g/Z)^j = (\mathbf{I}^1/Z)^{\mathbf{n}}(\Phi(\mathbf{m})/Z) \quad \text{for all} \\ \mathbf{n} \in \mathbf{N}_g \text{ such that } \mathbf{n}(g) \geq k, \text{ for all integers } j \geq 0, \text{ and for all} \\ \mathbf{m} \in \mathbf{N}_f. \quad (\%)$$

Now it is clear that $(\mathbf{I}^{\mathbf{n}}\Phi(\mathbf{m}) + Z)/Z$ equals the right-hand side of (%), so it remains to show that if j is large, then $Z \subseteq \mathbf{I}^{\mathbf{n}+j\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^j$ and that modulo Z this ideal is the left-hand side of (%). For this, since $Z = (0) : I_g^j$ for all large j , it follows that $Z \subseteq \mathbf{I}^{\mathbf{n}+j\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^j$. Also, it is readily checked that $(\mathbf{I}^{\mathbf{n}+j\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^j)/Z \subseteq (\mathbf{I}^1/Z)^{\mathbf{n}+j\mathbf{e}_j}(\Phi(\mathbf{m})/Z) : (I_g/Z)^j$, so let $x + Z$ be an element in this latter ideal. Now (%) shows that it may be assumed that j is large, and it follows that $xI_g^j \subseteq \mathbf{I}^{\mathbf{n}+j\mathbf{e}_g}\Phi(\mathbf{m}) + Z$, so $xI_g^{j+h} \subseteq \mathbf{I}^{\mathbf{n}+(j+h)\mathbf{e}_g}\Phi(\mathbf{m}) + I_g^h Z = \mathbf{I}^{\mathbf{n}+(j+h)\mathbf{e}_g}\Phi(\mathbf{m})$ for all large h . Therefore $x \in \mathbf{I}^{\mathbf{n}+(j+h)\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^{j+h} = \mathbf{I}^{\mathbf{n}+j\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^j$, since j is large, so $x + Z \in (\mathbf{I}^{\mathbf{n}+j\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^j)/Z$, hence $(\mathbf{I}^{\mathbf{n}+j\mathbf{e}_g}\Phi(\mathbf{m}) : I_g^j)/Z = (\mathbf{I}^1/Z)^{\mathbf{n}+j\mathbf{e}_j}(\Phi(\mathbf{m})/Z) : (I_g/Z)^j$.

The proof of (2.4.2) is similar, so it is omitted, and (2.4.3) is the special case $\mathbf{m} = 0 \in \mathbf{N}_f$ of (2.4.2). Q.E.D.

The following corollary of (2.3) verifies the statement in the first paragraph of this section.

(2.5) COROLLARY. *If H and I are ideals in a Noetherian ring R such that I is not nilpotent, and if $Z = \cup \{(0) : I^j; j \geq 1\}$, then $H^m I^{n+j} : I^j = H^m I^n + Z$ for all large positive integers j and n and for all integers $m \geq 0$.*

Proof. This follows immediately from either (2.4.1) or (2.4.2). Q.E.D.

This section is closed with the following remark, which is related to (2.1).

(2.6) Remark. If $\Phi = (\phi_1, \dots, \phi_g)$ is a finite collection of Noetherian filtrations on a Noetherian ring R , then:

(2.6.1) Let Γ be an infinite collection of regular ideals of R and let J be a regular ideal of R such that $\text{Rad}(G) \subseteq \text{Rad}(J)$ for all $G \in \Gamma$. (For example, Γ could be $\{\phi(i)\}_{i \geq 1}$, where ϕ is a filtration on R such that $\phi(1)$ is regular, and J could be any ideal containing $\phi(i)$ for some i .) Then there exist G_1, \dots, G_n in Γ such that $G_1 \cdots G_n \Phi(\mathbf{m}) : J \subseteq \Phi(\mathbf{m})$ for all $\mathbf{m} \in \mathbf{N}_g$.

(2.6.2) If $\phi_i(1)$ is regular for $i = 1, \dots, g$, if $\mathbf{j} \in \mathbf{P}_g$, and if H is an ideal of R , then $H\Phi(\mathbf{n}) : \Phi(\mathbf{j}) \subseteq H$ for all large $\mathbf{n} \in \mathbf{P}_g$.

Proof. For (2.6.1) let $\mathbf{R} = \mathbf{R}(R, \phi_1, \dots, \phi_g)$ and $\mathbf{u} = (u_1, \dots, u_g)$. Then \mathbf{R} is Noetherian, since each ϕ_i is, so it follows much as in the proof of (2.1.1)–(2.1.3) that there exist $G_1, \dots, G_n \in \Gamma$ such that $\mathbf{u}^m G_1 \cdots G_n \mathbf{R} : J\mathbf{R} =$

$\mathbf{u}^{\mathbf{m}}(G_1 \cdots G_n \mathbf{R} : J\mathbf{R}) \subseteq \mathbf{u}^{\mathbf{m}}\mathbf{R}$ for all $\mathbf{m} \in \mathbf{N}_g$, so the conclusion follows by contracting the first and third of these ideals to R .

For (2.6.2), since each ϕ_i is Noetherian, it is shown in [6, (2.4.3)] that for each $i = 1, \dots, g$ there exists a positive integer h_i such that $\phi_i(n_i + h_i) = \phi_i(h_i)\phi_i(n_i)$ for all $n_i \geq h_i$. It follows from this that if $\mathbf{h} = (h_1, \dots, h_g)$, then $\Phi(n\mathbf{h}) = (\Phi(\mathbf{h}))^n$ for all positive integers n . Also, if $\phi_i(1)$ is regular, then so is $\phi_i(k)$ for all $k \geq 1$, since $\text{Rad}(\phi_i(k)) = \text{Rad}(\phi_i(1))$. Therefore (2.2.1) shows that if n is large, then $H\Phi(n\mathbf{h}) : \Phi(\mathbf{j}) \subseteq H$. Statement (2.6.2) follows from this, since if $\mathbf{k} \geq \mathbf{m}$, then $\Phi(\mathbf{k}) \subseteq \Phi(\mathbf{m})$. Q.E.D.

Concerning (2.6.1), note that if $J \in \Gamma$, then (2.3.2) (with $g = 1$ and $I_1 = J$) shows that by taking each G_i to be J the conclusion can be sharpened to $J^{n+j}\Phi(\mathbf{m}) : J^j = J^n\Phi(\mathbf{m})$ for all $\mathbf{m} \in \mathbf{N}_g$, for all large n , and for all $j \geq 0$.

3. THE THEOREM OF SAKUMA AND OKUYAMA

In [9] M. Sakuma and H. Okuyama proved that if I is an ideal in an analytically unramified semi-local ring, then there exists a positive integer m (resp. k) such that $(I^{n+m})_a \subseteq I^n$ (resp. $(I^{n+k})_a = I^n(I^k)_a$) for all integers $n \geq 0$. (Also, see the comment at the start of Section 5.) In this section we generalize this result by proving the following theorem (see (3.2) for the definitions).

(3.1) THEOREM. *Let R be a locally analytically unramified Noetherian ring whose integral closure is a finite R -module, let $\phi \rightarrow \phi_x$ be a semi-prime operation on the set of filtrations ϕ on R such that $\phi_x \leq \phi_w$, let $\Phi = (\phi_1, \dots, \phi_f)$ be a finite collection of Noetherian filtrations on R , and let $\mathbf{I} = (I_1, \dots, I_g)$ be a finite collection of ideals of R . Then:*

(3.1.1) *For all large $\mathbf{m} \in \mathbf{P}_f$ it holds that $(\Phi(\mathbf{n} + \mathbf{m}))_x \subseteq \Phi(\mathbf{n})$ for all $\mathbf{n} \in \mathbf{N}_f$.*

(3.1.2) *There exists $\mathbf{k} \in \mathbf{P}_f$ such that $(\Phi(\mathbf{n} + \mathbf{k}))_x = \Phi(\mathbf{k})(\Phi(\mathbf{n}))_x$ for all $\mathbf{n} \geq \mathbf{k}$.*

(3.1.3) *For all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $(\mathbf{I}^{n+\mathbf{k}}\Phi(\mathbf{m}))_x = \mathbf{I}^n(\mathbf{I}^{\mathbf{k}}\Phi(\mathbf{m}))_x$ for all $\mathbf{n} \in \mathbf{N}_g$ and $\mathbf{m} \in \mathbf{N}_f$.*

(3.1.4) *For all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $(\mathbf{I}^{n+\mathbf{k}})_x = \mathbf{I}^n(\mathbf{I}^{\mathbf{k}})_x$ for all $\mathbf{n} \in \mathbf{N}_g$, so if \mathbf{k} is large and $\mathbf{H} = \mathbf{I}^{\mathbf{k}}$, then $(\mathbf{H}^n)_x = \mathbf{H}^n$ for all $\mathbf{n} \in \mathbf{N}_g$.*

To prove (3.1) we need several definitions and some facts concerning them, so we begin with these.

(3.2) DEFINITION. If R is a ring, then:

(3.2.1) The *integral closure* ϕ_a of a filtration ϕ on R is the sequence of ideals $\phi_a = \{(\phi(n))_a\}_{n \geq 0}$, where $(\phi(n))_a$ is the *integral closure in R of $\phi(n)$* ; therefore $(\phi(n))_a = \{x \in R; x \text{ satisfies an equation of the form } x^k + b_1 x^{k-1} + \dots + b_k = 0, \text{ where } b_i \in (\phi(n))^i \text{ for } i = 1, \dots, k\}$ (see (3.3.1)).

(3.2.2) The *weak integral closure* ϕ_w of a filtration ϕ on R is the sequence of ideals $\phi_w = \{(\phi(n))_w\}_{n \geq 0}$, where $(\phi(n))_w$ is the *weak integral closure of $\phi(n)$* ; therefore $(\phi(n))_w = \{x \in R; x \text{ satisfies an equation of the form } x^k + b_1 x^{k-1} + \dots + b_k = 0, \text{ where } b_i \in \phi(ni) \text{ for } i = 1, \dots, k\}$ (see (3.3.1)). If $\Phi = (\phi_1, \dots, \phi_g)$ is a finite collection of filtrations on R and $\mathbf{n} \in \mathbb{N}_g$, then we use $(\Phi(\mathbf{n}))_w$ to denote the weak integral closure of the ideal $\Phi(\mathbf{n})$ with respect to the filtration $\{\phi_1(in(1)) \cdots \phi_g(in(g))\}_{i \geq 0}$.

(3.2.3) Let Δ be a multiplicatively closed set of nonzero ideals of R and let H be an ideal of R . Then the Δ -closure H_Δ of H is defined by $H_\Delta = \cup \{HK : K \in \Delta\}$, and H is Δ -closed in case $H = H_\Delta$. Also, if ϕ is a filtration on R , then the Δ -closure ϕ_Δ of ϕ is the sequence of ideals $\phi_\Delta = \{(\phi(n))_\Delta\}_{n \geq 0}$, and ϕ is Δ -closed in case $\phi = \phi_\Delta$ (see (3.3.3)).

(3.2.4) Let (S, \leq) be a partially ordered semi-group. Then a mapping $H \rightarrow H_x$ on the elements H of S is a *semi-prime* operation in case for all $H, I \in S$, it holds that $H \leq H_x$; $H \leq I$ implies that $H_x \leq I_x$; $(H_x)_x = H_x$; and, $H_x I_x \leq (HI)_x$.

Concerning (3.2.1) and (3.2.2), it should be noted that the weak integral closure $(\phi(n))_w$ of $\phi(n)$ is called the integral closure of ϕ in [8]. This is certainly appropriate terminology, but in much of the older literature ϕ_w is called the weak integral closure of ϕ and the filtration ϕ_a is called the integral closure of ϕ , so it was decided to stay with the older terminology.

Concerning (3.2.3), note that if $\mathbf{I} = (I_1, \dots, I_g)$ is a finite collection of regular ideals of R and if $\Delta = \{\mathbf{I}^{\mathbf{n}}; \mathbf{n} \in \mathbb{N}_g\}$, then (2.3.2) shows that for every finite collection $\Phi = (\phi_1, \dots, \phi_f)$ of Noetherian filtrations on R the ideals $\mathbf{I}^{\mathbf{n}}\Phi(\mathbf{m})$ are Δ -closed for all large $\mathbf{n} \in \mathbb{P}_g$ and for all $\mathbf{m} \in \mathbb{N}_f$.

(3.3) Remark. If R is a ring, then:

(3.3.1) If ϕ is a filtration on R , then it is shown in [4, (4.2.1) and (2.2)] that ϕ_a and ϕ_w are filtrations on R such that $\phi \leq \phi_a \leq \phi_w$. (In general, if ϕ and γ are filtrations on R , then $\phi \leq \gamma$ means that $\phi(n) \subseteq \gamma(n)$ for all $n \geq 0$.) Also, if h is an integer such that $\phi(nh) = (\phi(h))^n$ for all $n \geq 1$, then it follows from (3.2.1) and (3.2.2) that $(\phi(nh))_a = (\phi(nh))_w$ for all $n \geq 1$.

(3.3.2) It is shown in [4, (4.1)] that if $I \rightarrow I_x$ is a semi-prime operation on the set of ideals I of R , then $\phi \rightarrow \phi_x = \{(\phi(n))_x\}_{n \geq 0}$ is a semi-prime

operation on the set of filtrations ϕ of R . Also, it is shown in [5] (resp. [4, (2.4)]) that $I \rightarrow I_a$ (resp. $\phi \rightarrow \phi_w$) is a semi-prime operation on the set of ideals I (resp. filtrations ϕ) of R .

(3.3.3) Let \mathcal{A} be a multiplicatively closed set of nonzero ideals of R . Then it is readily checked that if I is an ideal of R and K_1 and K_2 are in \mathcal{A} , then both $IK_1 : K_1$ and $IK_2 : K_2$ are contained in $IK_1K_2 : K_1K_2$, so it follows that $I_{\mathcal{A}}$ is an ideal in R . Also, it is shown in [7, (2.4)] that $I \rightarrow I_{\mathcal{A}}$ is a semi-prime operation on the set of ideals I of R , so it follows from (3.3.2) that $\phi \rightarrow \phi_{\mathcal{A}}$ is a semi-prime operation on the set of filtrations ϕ on R .

(3.3.4) Let \mathcal{A} be a multiplicatively closed set of nonzero ideals of R . Then it is shown in [7, (3.2)] that if $\text{height}(K) \geq 1$ for all ideals K in \mathcal{A} , then for all ideals I of R it holds that $I_{\mathcal{A}} \subseteq I_a$. It follows from this that if ϕ is a filtration on R and if $\text{height}(K) \geq 1$ for all ideals $K \in \mathcal{A}$, then $\phi_{\mathcal{A}} \leq \phi_a$ (see (3.2.1), (3.3.1), and (3.3.3)).

Concerning (3.2.4), (3.3.2), and (3.3.3), it should be noted that there are many other types of semi-prime operations on the set of ideals I or R . For example, if A is an R -algebra and Γ is a multiplicatively closed set of nonzero ideals of R , then $I \rightarrow I_e = IA \cap R$ and $I \rightarrow I_{\Gamma} = \bigcup \{I : G; G \in \Gamma\}$ are semi-prime operations, and $I_e \subseteq I_a$ if A is integral over R . Also, if $\{I \rightarrow I_{x_i}; i \in \mathcal{A}\}$ is an arbitrary collection of semi-prime operations, then $I \rightarrow I_x = \bigcap I_{x_i}$ is also a semi-prime operation, and $I_x \subseteq I_a$ if $I_{x_i} \subseteq I_a$ for at least one i .

Proof of (3.1). It is shown in [6, (2.4.3)] that a filtration ϕ is Noetherian if and only if there exists a positive integer h such that $\phi(n+h) = \phi(n)\phi(h)$ for all integers $n \geq h$. Therefore for $i = 1, \dots, f$ let h_i be a positive integer such that $\phi_i(n_i + h_i) = \phi_i(n_i)\phi_i(h_i)$ for all $n_i \geq h_i$. Then it follows that $\mathbf{R} = \mathbf{R}(R, \phi_1, \dots, \phi_f)$ is generated over $R[u_1, \dots, u_f]$ by $\{t_i^{n_i}\phi_i(n_i); i = 1, \dots, f \text{ and } n_i = 1, \dots, 2h_i - 1\}$, so \mathbf{R} is Noetherian.

For (3.1.1) it follows as in [1, Lemma 1] that the hypotheses on R imply that the integral closure \mathbf{R}' of \mathbf{R} is a finite \mathbf{R} -module. Further, if $\mathbf{u} = (u_1, \dots, u_f)$, then $\mathbf{u}^n \mathbf{R}' \cap R = (\Phi(\mathbf{n}))_w$ for all $\mathbf{n} \in \mathbf{N}_f$, so the hypothesis that $\phi_x \leq \phi_w$ shows that $\mathbf{R} \subseteq \mathbf{B} \subseteq \mathbf{R}'$, where $\mathbf{B} = R[u_1, \dots, u_f, \{t^n(\Phi(\mathbf{n}))_x; \mathbf{n} \in \mathbf{N}_f\}]$. Therefore \mathbf{B} is a graded finite \mathbf{R} -module, since \mathbf{R} is Noetherian, and $\mathbf{R}[1/\mathbf{u}^1] = R[u_1, \dots, u_f, t_1, \dots, t_f] = \mathbf{B}[1/\mathbf{u}^1]$, so it follows that for all large $\mathbf{m} \in \mathbf{P}_f$ it holds that $\mathbf{u}^{n+\mathbf{m}}\mathbf{B} \subseteq \mathbf{u}^n\mathbf{R}$ for all $\mathbf{n} \in \mathbf{N}_f$. Therefore $(\Phi(\mathbf{n} + \mathbf{m}))_x = \mathbf{u}^{n+\mathbf{m}}\mathbf{B} \cap R \subseteq \mathbf{u}^n\mathbf{R} \cap R = \Phi(\mathbf{n})$ for all $\mathbf{n} \in \mathbf{N}_f$, so (3.1.1) holds.

For (3.1.2) let j_i be the least common multiple of $2, 3, \dots, 2h_i - 1$, and for $l = 1, \dots, 2h_i - 1$ let $m_{i,l}$ be the positive integer such that $lm_{i,l} = j_i$. Then $(t_i^l \phi_i(l))^{m_{i,l}} \subseteq t_i^{j_i} \phi_i(j_i)$, so it follows from the last sentence in the first paragraph of this proof that \mathbf{R} is integral and finitely generated over its

graded subring $A = R[u_1^{j_1}, \dots, u_f^{j_f}, t_1^{j_1}\phi_1(j_1), \dots, t_f^{j_f}\phi_f(j_f)]$. Also, as noted in the preceding paragraph, B is a graded finite R -module, hence B is a graded finite A -module.

Let $\Theta_1, \dots, \Theta_m$ be homogeneous elements of B that are a linear basis for B over A , let $\deg_i(\Theta_l) = d_{i,l}$ (so $\Theta_l \in (\Phi(\mathbf{d}_l))_x$ for $l = 1, \dots, m$, where $\mathbf{d}_l = (d_{1,l}, \dots, d_{f,l})$), and let \mathbf{d} be an element in P_f such that $\mathbf{d} \geq \mathbf{d}_l + 1$ for $l = 1, \dots, m$. Let $\mathbf{n} \geq \mathbf{d}$ and let y be an element in $(\Phi(\mathbf{n}))_x$. Then $y\mathbf{t}^{\mathbf{n}} \in B$, so $y\mathbf{t}^{\mathbf{n}} = \sum_{l=1}^m b_l \Theta_l$ for some homogeneous elements b_l (necessarily either zero or nonzero of degree $\mathbf{n} - \mathbf{d}_l$) in A . By resubscripting, if necessary, assume that $b_l \neq 0$ for $l = 1, \dots, m' \leq m$. Then $\mathbf{n} - \mathbf{d}_l \geq \mathbf{1}$, and it follows from the definition of A that its homogeneous elements of positive degree have degree a multiple $\mathbf{q} \in N_f$ of $\mathbf{j} = (j_1, \dots, j_f)$, so for $l = 1, \dots, m'$ there exists $\mathbf{q}_l \in P_f$ such that $\mathbf{j}\mathbf{q}_l = \mathbf{n} - \mathbf{d}_l$. Therefore it follows that $y \in \sum_{l=1}^{m'} (\Phi(\mathbf{j}))^{q_l} (\Phi(\mathbf{d}_l))_x = \Phi(\mathbf{j})(\sum_{l=1}^{m'} (\Phi(\mathbf{j}))^{q_l-1} (\Phi(\mathbf{d}_l))_x) \subseteq \Phi(\mathbf{j})(\Phi(\mathbf{n} - \mathbf{j}))_x$, since $(\Phi(\mathbf{j}))^{q_l-1} (\Phi(\mathbf{d}_l))_x \subseteq (\Phi(\mathbf{j}(\mathbf{q}_l - 1) + \mathbf{d}_l))_x$ and $\mathbf{j}(\mathbf{q}_l - 1) + \mathbf{d}_l = \mathbf{n} - \mathbf{j}$. It follows that $(\Phi(\mathbf{n}))_x \subseteq \Phi(\mathbf{j})(\Phi(\mathbf{n} - \mathbf{j}))_x$, and the opposite inclusion is clear, so

$$(\Phi(\mathbf{n}))_x = \Phi(\mathbf{j})(\Phi(\mathbf{n} - \mathbf{j}))_x \quad \text{for all } \mathbf{n} \geq \mathbf{d}. \quad (*)$$

Therefore let $\mathbf{k} = \mathbf{j}\mathbf{d}$ and let $\mathbf{n} \geq \mathbf{k}$. Then it follows from $(*)$ that $(\Phi(\mathbf{k} + \mathbf{n}))_x = (\Phi(\mathbf{j}\mathbf{d} + \mathbf{n}))_x = \Phi(\mathbf{j})(\Phi(\mathbf{k} + \mathbf{n} - \mathbf{j}))_x = \dots = (\Phi(\mathbf{j}))^{\mathbf{d}} (\Phi(\mathbf{n}))_x \subseteq \Phi(\mathbf{j}\mathbf{d})(\Phi(\mathbf{n}))_x = \Phi(\mathbf{k})(\Phi(\mathbf{n}))_x \subseteq (\Phi(\mathbf{k} + \mathbf{n}))_x$, so $(\Phi(\mathbf{k} + \mathbf{n}))_x = \Phi(\mathbf{k})(\Phi(\mathbf{n}))_x$ for all $\mathbf{n} \geq \mathbf{k}$, so (3.1.2) holds.

For (3.1.3) let $\gamma_i = \{I_i^n\}_{n \geq 0}$ for $i = 1, \dots, g$, let $\gamma_{g+j} = \phi_j$ for $j = 1, \dots, f$, and let $\Gamma = (\gamma_1, \dots, \gamma_{g+f})$. Also, let $C = R(R, \gamma_1, \dots, \gamma_{g+f})$ and $D = R[u_1, \dots, u_{g+f}, \{\mathbf{t}^{\mathbf{n}}(\Phi(\mathbf{n}))_x; \mathbf{n} \in N_{g+f}\}]$. Then it follows as in the second paragraph of this proof that D is a finite Z_{g+f} -graded C -module. Let $\theta_1, \dots, \theta_p$ be elements in D that are a linear basis for D over C , let $\deg_i(\theta_l) = d_{i,l}$ (so $\theta_l \in (\Gamma(\mathbf{d}_l))_x$ for $l = 1, \dots, p$, where $\mathbf{d}_l = (d_{1,l}, \dots, d_{g+f,l})$), and let $\mathbf{d} \in N_{g+f}$ such that $\mathbf{d}(i) \geq \mathbf{d}_l(i)$ for $i = 1, \dots, g$ (the $\mathbf{d}(j)$ are arbitrary nonnegative integers for $j = g+1, \dots, g+f$). Let $\mathbf{n} \geq \mathbf{d}$ and let $y \in (\Gamma(\mathbf{n}))_x$. Then $y\mathbf{t}^{\mathbf{n}} \in D$, so $y\mathbf{t}^{\mathbf{n}} = \sum_{l=1}^p b_l \theta_l$ for some homogeneous elements b_l (either zero or nonzero of degree $\mathbf{n} - \mathbf{d}_l$) in C . Therefore it follows that $y \in \sum_{l=1}^p \Gamma(\mathbf{n} - \mathbf{d}_l)(\Gamma(\mathbf{d}_l))_x$. Now for $\mathbf{m} \in N_{g+f}$ define $\mathbf{m}' \in N_g$ (resp. $\mathbf{m}'' \in N_f$) by $\mathbf{m}'(i) = \mathbf{m}(i)$ for $i = 1, \dots, g$ (resp. $\mathbf{m}''(j) = \mathbf{m}(g+j)$ for $j = 1, \dots, f$), and note that $\mathbf{n}(i) - \mathbf{d}_l(i) \geq \mathbf{n}(i) - \mathbf{d}(i) \geq 0$ for $i = 1, \dots, g$ and that $\Gamma(\mathbf{n} - \mathbf{d}_l) = \mathbf{I}^{\mathbf{n}' - \mathbf{d}_l'} \Phi(\mathbf{n}'' - \mathbf{d}_l'') = \mathbf{I}^{\mathbf{n}' - \mathbf{d}_l'} \mathbf{I}^{\mathbf{d}' - \mathbf{d}_l'} \Phi(\mathbf{n}'' - \mathbf{d}_l'')$ for $i = 1, \dots, p$. Therefore $y \in \sum_{l=1}^p \mathbf{I}^{\mathbf{n}' - \mathbf{d}_l'} \mathbf{I}^{\mathbf{d}' - \mathbf{d}_l'} \Phi(\mathbf{n}'' - \mathbf{d}_l'') (\mathbf{I}^{\mathbf{d}_l'} \Phi(\mathbf{d}_l''))_x \subseteq \mathbf{I}^{\mathbf{n}' - \mathbf{d}'} (\mathbf{I}^{\mathbf{d}'} \Phi(\mathbf{n}''))_x$, and hence it follows that $(\Gamma(\mathbf{n}))_x = \mathbf{I}^{\mathbf{n}' - \mathbf{d}'} (\mathbf{I}^{\mathbf{d}'} \Phi(\mathbf{n}''))_x$. Statement (3.1.3) readily follows from this by letting $\mathbf{k} \geq (\mathbf{d}(1), \dots, \mathbf{d}(g))$.

Statement (3.1.4) is the special case where $\mathbf{m} = 0 \in N_g$ of (3.1.3). Q.E.D.

Corollary (3.4) gives three important special cases of (3.1).

(3.4) COROLLARY. Let R , $\Phi = (\phi_1, \dots, \phi_f)$, and $I = (I_1, \dots, I_g)$ be as in (3.1), and let Δ be a multiplicatively closed set of regular ideals of R . Then:

(3.4.1) For all large $\mathbf{m} \in \mathbf{P}_f$ it holds that $(\Phi(\mathbf{n} + \mathbf{m}))_w \subseteq \Phi(\mathbf{n})$, $(\Phi(\mathbf{n} + \mathbf{m}))_a \subseteq \Phi(\mathbf{n})$, and $(\Phi(\mathbf{n} + \mathbf{m}))_\Delta \subseteq \Phi(\mathbf{n})$ for all $\mathbf{n} \in \mathbf{N}_f$.

(3.4.2) There exists $\mathbf{k} \in \mathbf{P}_f$ such that $(\Phi(\mathbf{n} + \mathbf{k}))_w = \Phi(\mathbf{k})(\Phi(\mathbf{n}))_w$, $(\Phi(\mathbf{n} + \mathbf{k}))_a = \Phi(\mathbf{k})(\Phi(\mathbf{n}))_a$, and $(\Phi(\mathbf{n} + \mathbf{k}))_\Delta = \Phi(\mathbf{k})(\Phi(\mathbf{n}))_\Delta$ for all $\mathbf{n} \geq \mathbf{k}$.

(3.4.3) For all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $(\mathbf{I}^{\mathbf{n} + \mathbf{k}}\Phi(\mathbf{m}))_w = \mathbf{I}^{\mathbf{n}}(\mathbf{I}^{\mathbf{k}}\Phi(\mathbf{m}))_w$, $(\mathbf{I}^{\mathbf{n} + \mathbf{k}}\Phi(\mathbf{m}))_a = \mathbf{I}^{\mathbf{n}}(\mathbf{I}^{\mathbf{k}}\Phi(\mathbf{m}))_a$, and $(\mathbf{I}^{\mathbf{n} + \mathbf{k}}\Phi(\mathbf{m}))_\Delta = \mathbf{I}^{\mathbf{n}}(\mathbf{I}^{\mathbf{k}}\Phi(\mathbf{m}))_\Delta$ for all $\mathbf{n} \in \mathbf{N}_g$ and $\mathbf{m} \in \mathbf{N}_f$.

Proof. It is noted in (3.3.2) that $\phi \rightarrow \phi_w$ and $\phi \rightarrow \phi_a$ are semi-prime operations on the set of filtrations ϕ on R , and (3.3.3) shows that this also holds for $\phi \rightarrow \phi_\Delta$. Also, (3.3.1) shows that $\phi_a \leq \phi_w$, and since the ideals in Δ are regular, (3.3.4) shows that $\phi_\Delta \leq \phi_a$. Therefore all three parts follow immediately from (3.1). Q.E.D.

Corollary (3.5) is a variation of (3.1.2).

(3.5) COROLLARY. Let R and $\phi \rightarrow \phi_x$ be as in (3.1) and let $\Phi = (\phi_1, \dots, \phi_g)$ be a finite collection of Noetherian filtrations on R . Then there exists $\mathbf{h} \in \mathbf{P}_g$ such that $(\Phi((\mathbf{n} + \mathbf{k})\mathbf{h} + \mathbf{r}))_x = (\Phi(\mathbf{h}))^{\mathbf{n}}(\Phi(\mathbf{k}\mathbf{h} + \mathbf{r}))_x$ for all large $\mathbf{k} \in \mathbf{P}_g$, for all $\mathbf{n} \in \mathbf{N}_g$, and for all $\mathbf{r} \in \mathbf{N}_g$ such that $\mathbf{h}(i) \leq \mathbf{r}(i) < 2\mathbf{h}(i)$ for $i = 1, \dots, g$.

Proof. As noted at the start of the proof of (3.1), for $i = 1, \dots, g$ there exists a positive integer h_i such that $\phi_i(n_i + h_i) = \phi_i(n_i)\phi_i(h_i)$ for all $n_i \geq h_i$. Let $\mathbf{h} = (h_1, \dots, h_g)$, so $\Phi(\mathbf{n}\mathbf{h} + \mathbf{r}) = (\Phi(\mathbf{h}))^{\mathbf{n}}\Phi(\mathbf{r})$ for all $\mathbf{n} \in \mathbf{N}_g$ and for all \mathbf{r} such that $\mathbf{h}(i) \leq \mathbf{r}(i) < 2\mathbf{h}(i)$ for $i = 1, \dots, g$. Also, it follows from (3.1.3) that for each such \mathbf{r} the following holds for all large $\mathbf{k} \in \mathbf{P}_g$: $((\Phi(\mathbf{h}))^{\mathbf{n} + \mathbf{k}}\Phi(\mathbf{r}))_x = (\Phi(\mathbf{h}))^{\mathbf{n}}((\Phi(\mathbf{h}))^{\mathbf{k}}\Phi(\mathbf{r}))_x$. The conclusion clearly follows from this, since $(\Phi(\mathbf{h}))^{\mathbf{n} + \mathbf{k}}\Phi(\mathbf{r}) = \Phi((\mathbf{n} + \mathbf{k})\mathbf{h} + \mathbf{r})$ and $(\Phi(\mathbf{h}))^{\mathbf{k}}\Phi(\mathbf{r}) = \Phi(\mathbf{k}\mathbf{h} + \mathbf{r})$. Q.E.D.

In closing this section, it should be noted that (3.5) holds when ϕ_x is any one of ϕ_w , ϕ_a , and, ϕ_Δ , where Δ is a multiplicatively closed set of regular ideals of R .

4. AN EXTENSION OF (3.1) TO ALL NOETHERIAN RINGS

Our goal in this section is to show that an analog of (3.1) holds in all Noetherian rings. Specifically, we prove the following theorem.

(4.1) THEOREM. Let R be a Noetherian ring, let Δ be a finitely generated multiplicatively closed set of regular ideals of R , let $\phi \rightarrow \phi_x$ be a semi-prime

operation on the set of filtrations ϕ on R such that $\phi_x \leq \phi_\Delta$, let $\Phi = (\phi_1, \dots, \phi_f)$ be a finite collection of Noetherian filtrations on R , and let $\mathbf{I} = (I_1, \dots, I_g)$ be a finite collection of ideals of R . Then:

(4.1.1) For all large $\mathbf{m} \in \mathbf{P}_f$ it holds that $(\Phi(\mathbf{n} + \mathbf{m}))_x \subseteq \Phi(\mathbf{n})$ for all $\mathbf{n} \in \mathbf{N}_f$.

(4.1.2) There exists $\mathbf{k} \in \mathbf{P}_f$ such that $(\Phi(\mathbf{n} + \mathbf{k}))_x = \Phi(\mathbf{k})(\Phi(\mathbf{n}))_x$ for all $\mathbf{n} \geq \mathbf{k}$.

(4.1.3) For all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $(\mathbf{I}^{\mathbf{n} + \mathbf{k}}\Phi(\mathbf{m}))_x = \mathbf{I}^{\mathbf{n}}(\mathbf{I}^{\mathbf{k}}\Phi(\mathbf{m}))_x$ for all $\mathbf{n} \in \mathbf{N}_g$ and $\mathbf{m} \in \mathbf{N}_f$.

(4.1.4) For all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $(\mathbf{I}^{\mathbf{n} + \mathbf{k}})_x = \mathbf{I}^{\mathbf{n}}(\mathbf{I}^{\mathbf{k}})_x$ for all $\mathbf{n} \in \mathbf{N}_g$, so if \mathbf{k} is large and $\mathbf{H} = \mathbf{I}^{\mathbf{k}}$, then $(\mathbf{H}^{\mathbf{n}})_x = \mathbf{H}^{\mathbf{n}}$ for all $\mathbf{n} \in \mathbf{N}_g$.

To prove (4.1) we need two preliminary results, both of which are of some interest in themselves.

(4.2) THEOREM. Let $\phi \rightarrow \phi_x$ be a semi-prime operation on the set of filtrations ϕ on a Noetherian ring R , let $\Phi = (\phi_1, \dots, \phi_g)$ be a finite collection of Noetherian filtrations on R , let $\mathbf{R} = \mathbf{R}(R, \phi_1, \dots, \phi_g)$, and let $\mathbf{D} = R[u_1, \dots, u_g, \{\mathbf{t}^{\mathbf{n}}(\Phi(\mathbf{n}))_x; \mathbf{n} \in \mathbf{N}_g\}]$. Then the following are equivalent:

(4.2.1) \mathbf{D} is a finite \mathbf{R} -module.

(4.2.2) There exists $\mathbf{m} \in \mathbf{N}_g$ such that $\mathbf{u}^{\mathbf{m}}\mathbf{D} \subseteq \mathbf{R}$.

(4.2.3) For all large $\mathbf{m} \in \mathbf{P}_g$ it holds that $\mathbf{u}^{\mathbf{n} + \mathbf{m}}\mathbf{D} \subseteq \mathbf{u}^{\mathbf{n}}\mathbf{R}$ for all $\mathbf{n} \in \mathbf{N}_g$.

(4.2.4) For all large $\mathbf{m} \in \mathbf{P}_g$ it holds that $(\Phi(\mathbf{n} + \mathbf{m}))_x \subseteq \Phi(\mathbf{n})$ for all $\mathbf{n} \in \mathbf{N}_g$.

(4.2.5) If $\Gamma = (\phi_1, \dots, \phi_h)$ is an arbitrary nonempty subset of Φ , then \mathbf{B} is a finite \mathbf{A} -module, where $\mathbf{A} = \mathbf{R}(R, \phi_1, \dots, \phi_h)$ and $\mathbf{B} = R[u_1, \dots, u_h, \{\mathbf{t}^{\mathbf{n}}(\Gamma(\mathbf{n}))_x; \mathbf{n} \in \mathbf{N}_h\}]$.

(4.2.6) If $\Gamma = (\phi_1, \dots, \phi_h)$ is an arbitrary nonempty subsets of Φ , then for all large $\mathbf{m} \in \mathbf{P}_h$ it holds that $(\Gamma(\mathbf{n} + \mathbf{m}))_x \subseteq \Gamma(\mathbf{n})$ for all $\mathbf{n} \in \mathbf{N}_h$.

Proof. (4.2.2) implies that $\mathbf{D} \subseteq (1/\mathbf{u}^{\mathbf{m}})\mathbf{R}$, so (4.2.2) \Rightarrow (4.2.1) since $\mathbf{u}^{\mathbf{m}}$ is a regular element in \mathbf{R} .

The last part of the second paragraph of the proof of (3.1) essentially shows that (4.2.1) \Rightarrow (4.2.4).

To show that (4.2.4) \Rightarrow (4.2.3), since $\mathbf{u}^{\mathbf{n} + \mathbf{m}}\mathbf{D}$ and $\mathbf{u}^{\mathbf{n}}\mathbf{R}$ are homogeneous ideals, it suffices to show that if $\mathbf{m} \in \mathbf{P}_g$ is large, if $\mathbf{n} \in \mathbf{N}_g$, and if $b\mathbf{t}^{\mathbf{i}} \in \mathbf{u}^{\mathbf{n} + \mathbf{m}}\mathbf{D}$ (with $b \in R$), then $b\mathbf{t}^{\mathbf{i}} \in \mathbf{u}^{\mathbf{n}}\mathbf{R}$. For this, it follows that $b\mathbf{t}^{\mathbf{i} + \mathbf{n} + \mathbf{m}} \in \mathbf{D}$, so $b \in \mathbf{u}^{\mathbf{i} + \mathbf{n} + \mathbf{m}}\mathbf{D} \cap R = (\Phi(\mathbf{i} + \mathbf{n} + \mathbf{m}))_x \subseteq \Phi(\mathbf{i} + \mathbf{n})$, by (4.2.4). Therefore $b\mathbf{t}^{\mathbf{i} + \mathbf{n}} \in \mathbf{R}$, so $b\mathbf{t}^{\mathbf{i}} \in \mathbf{u}^{\mathbf{n}}\mathbf{R}$, as desired.

It is clear that (4.2.3) \Rightarrow (4.2.2).

Therefore (4.2.1)–(4.2.4) are equivalent, so it follows that (4.2.5) and (4.2.6) are equivalent for each fixed nonempty subset Γ of Φ , and hence (4.2.5) and (4.2.6) are equivalent.

It is clear that (4.2.6) \Rightarrow (4.2.4).

Finally, let $\Theta = (\phi_{h+1}, \dots, \phi_g)$ and assume that (4.2.5) does not hold. Let $\mathbf{s} = (t_{h+1}, \dots, t_g)$. Then $\mathbf{B}_1 = \mathbf{B}[u_{h+1}, \dots, u_g, \{\mathbf{s}^n \Theta(\mathbf{n}); \mathbf{n} \in \mathbf{N}_{g-h}\}]$ is not a finite \mathbf{A}_1 -module, where $\mathbf{A}_1 = \mathbf{A}[u_{h+1}, \dots, u_g, \{\mathbf{s}^n \Theta(\mathbf{n}); \mathbf{n} \in \mathbf{N}_{g-h}\}]$ (since otherwise $\mathbf{B}[u_{h+1}, \dots, u_g, t_{h+1}, \dots, t_g] = \mathbf{B}_1[1/(u_{h+1} \cdots u_g)]$ is a finite $\mathbf{A}_1[1/(u_{h+1} \cdots u_g)] = \mathbf{A}[u_{h+1}, \dots, u_g, t_{h+1}, \dots, t_g]$ -module, and hence it follows that \mathbf{B} is a finite \mathbf{A} -module, and this contradicts the assumption that (4.2.5) does not hold). However, $\mathbf{A}_1 = \mathbf{R}$ and $\mathbf{B}_1 \subseteq \mathbf{D}$, and this implies that (4.2.1) does not hold. Therefore (4.2.1) \Rightarrow (4.2.5). Q.E.D.

The following remark lists three statements that, in special cases, are equivalent to those in (4.2).

(4.3) *Remark.* With the notation of (4.2):

(4.3.1) Assume that ϕ_i is the sequence of powers of an ideal I_i for $i = 1, \dots, g$ and let $\mathbf{I} = (I_1, \dots, I_g)$. Then (4.2.1)–(4.2.6) are also equivalent to each of the following statements: (a) For all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $(\mathbf{I}^{\mathbf{n}+\mathbf{k}})_x = \mathbf{I}^{\mathbf{n}}(\mathbf{I}^{\mathbf{k}})_x$ for all $\mathbf{n} \in \mathbf{N}_g$, and (b) for each nonempty subset $\mathbf{J} = (I_1, \dots, I_h)$ of \mathbf{I} and for all large $\mathbf{k} \in \mathbf{P}_h$ it holds that $(\mathbf{J}^{\mathbf{n}+\mathbf{k}})_x = \mathbf{J}^{\mathbf{n}}(\mathbf{J}^{\mathbf{k}})_x$ for all $\mathbf{n} \in \mathbf{N}_h$.

(4.3.2) If there exists a multiplicatively closed set \mathcal{A} of regular ideals of R such that $\phi_x = \phi_{\mathcal{A}}$ for all filtrations ϕ on R , then (4.2.1)–(4.2.6) are also equivalent to: There exists $K \in \mathcal{A}$ such that $(\Phi(\mathbf{n}))_{\mathcal{A}} = \Phi(\mathbf{n})K : K$ for all $\mathbf{n} \in \mathbf{N}_g$.

Proof. For (4.3.1), (4.3.1a) \Rightarrow (4.2.4), since the hypothesis implies that $\Phi(\mathbf{i}) = \mathbf{I}^{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{N}_g$, and the proof of (3.1.3) essentially shows that (4.2.1) \Rightarrow (4.3.1a). Therefore (4.3.1a) is equivalent to (4.2.4) (since (4.2.1) is equivalent to (4.2.4)), so it follows that (4.3.1b) is equivalent to (4.2.6) for each fixed nonempty subset $\mathbf{J} (= \Gamma)$ of $\mathbf{I} (= \Phi)$, so (4.3.1b) is equivalent to (4.2.6). The conclusion now follows from the fact that (4.2.1)–(4.2.6) are equivalent.

For (4.3.2), if (4.2.1) holds, then there exists $\mathbf{m} \in \mathbf{P}_g$ such that $\mathbf{D} = \sum_{0 \leq \mathbf{i} \leq \mathbf{m}} (t^{\mathbf{i}}(\Phi(\mathbf{i}))_{\mathcal{A}})\mathbf{R}$. Now, since R is Noetherian, for each $\mathbf{i} \in \mathbf{N}_g$ such that $\mathbf{i} \leq \mathbf{m}$ there exists an ideal $K(\mathbf{i}) \in \mathcal{A}$ such that $(\Phi(\mathbf{i}))_{\mathcal{A}} = \Phi(\mathbf{i})K(\mathbf{i}) : K(\mathbf{i})$. Let $K = \prod_{0 \leq \mathbf{i} \leq \mathbf{m}} K(\mathbf{i})$, so for each $\mathbf{i} \leq \mathbf{m}$ it holds that $(\Phi(\mathbf{i}))_{\mathcal{A}} = \Phi(\mathbf{i})K : K$ (since $(\Phi(\mathbf{i}))_{\mathcal{A}} = \Phi(\mathbf{i})K(\mathbf{i}) : K(\mathbf{i}) \subseteq \Phi(\mathbf{i})K : K \subseteq (\Phi(\mathbf{i}))_{\mathcal{A}}$), and hence $\mathbf{D} = \sum_{0 \leq \mathbf{i} \leq \mathbf{m}} (t^{\mathbf{i}}(\Phi(\mathbf{i})K : K))\mathbf{R}$. Let $\mathbf{C} = \sum_{\mathbf{i} \in \mathbf{N}_g} (t^{\mathbf{i}}(\Phi(\mathbf{i})K : K))\mathbf{R}$. Then the preceding formula for \mathbf{D} shows that $\mathbf{D} \subseteq \mathbf{C}$. Also, \mathbf{C} is an \mathbf{R} -module, since $\Phi(\mathbf{i})(\Phi(\mathbf{j})K : K) \subseteq \Phi(\mathbf{i})\Phi(\mathbf{j})K : K \subseteq \Phi(\mathbf{i} + \mathbf{j})K : K$ for

all $i, j \in \mathbf{N}_g$, and $C \subseteq D$, since $(\Phi(i)K : K)t^i \subseteq ((\Phi(i))_A)t^i$ (= the i th homogeneous component of D) for all $i \in \mathbf{N}_g$. Therefore $C = D$, so by comparing the homogeneous components of D and C it follows that $(\Phi(\mathbf{n}))_A = \Phi(\mathbf{n})K : K$ for all $\mathbf{n} \in \mathbf{N}_g$, and hence (4.2.1) \Rightarrow (4.3.2). And if (4.3.2) holds, then $D = \sum_{i \in \mathbf{N}_g} (t^i(\Phi(i)K : K))\mathbf{R}$, so $KD \subseteq \mathbf{R}$. Therefore, since the hypothesis implies that K is regular, it follows that (4.3.2) \Rightarrow (4.2.1).
Q.E.D.

(4.4) THEOREM. Let $\mathbf{I} = (I_1, \dots, I_g)$ be a finite collection of regular ideals of a Noetherian ring R , let $\Phi = (\phi_1, \dots, \phi_f)$ be a finite collection of Noetherian filtrations on R , let $\mathbf{R} = \mathbf{R}(R, I_1, \dots, I_g, \phi_1, \dots, \phi_f)$, let $\Delta = \{\mathbf{I}^{\mathbf{n}}; \mathbf{n} \in \mathbf{N}_g\}$, and let $\mathbf{D} = R[u_1, \dots, u_{g+f}, \{t^{\mathbf{n}}(\mathbf{I}^{\mathbf{n}}\Phi(\mathbf{n}''))_A; \mathbf{n} \in \mathbf{N}_{g+f}, \mathbf{n}' \in \mathbf{N}_g \text{ is such that } \mathbf{n}'(i) = \mathbf{n}(i) \text{ for } i = 1, \dots, g, \text{ and } \mathbf{n}'' \in \mathbf{N}_f \text{ is such that } \mathbf{n}''(i) = \mathbf{n}(g+i) \text{ for } i = 1, \dots, f\}]\}$. Then \mathbf{D} is a finite \mathbf{R} -module.

Proof. For convenience of notation, for $\mathbf{n} \in \mathbf{N}_{g+f}$ define $\mathbf{n}' \in \mathbf{N}_g$ (resp. $\mathbf{n}'' \in \mathbf{N}_f$) by $\mathbf{n}'(i) = \mathbf{n}(i)$ for $i = 1, \dots, g$ (resp. $\mathbf{n}''(i) = \mathbf{n}(g+i)$ for $i = 1, \dots, f$). Then in (3.3.3) it is noted that $I \rightarrow I_A$ is a semi-prime operation on the set of ideals I of R , so $(\mathbf{I}'\Phi(\mathbf{i}''))_A(\mathbf{I}'\Phi(\mathbf{j}''))_A \subseteq (\mathbf{I}'^{i'+j'}\Phi(\mathbf{i}''+\mathbf{j}''))_A$ for all \mathbf{i}, \mathbf{j} in \mathbf{N}_{g+f} . Thus, in particular: (a) $(\mathbf{I}')_A(\mathbf{I}'\Phi(\mathbf{j}''))_A \subseteq (\mathbf{I}'^{i'+j'}\Phi(\mathbf{j}''))_A$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{N}_{g+f}$ such that $\mathbf{i}'' = 0$. Also, since each I_i is regular, it follows from (2.3.3) that: (b) $(\mathbf{I}'^{\mathbf{n}})_A = \mathbf{I}'^{\mathbf{n}'}$ for all $\mathbf{n} \in \mathbf{N}_{g+f}$ such that \mathbf{n}' is large and $\mathbf{n}'' = 0$. Statement (2.3.2) shows that: (c) $(\mathbf{I}^{\mathbf{n}}\Phi(\mathbf{n}''))_A = \mathbf{I}^{\mathbf{n}'}\Phi(\mathbf{n}'')$ for all $\mathbf{n} \in \mathbf{N}_{g+f}$ such that \mathbf{n}' is large. Therefore it follows that if $\mathbf{n} \in \mathbf{N}_{g+f}$ is such that \mathbf{n}' is large and $\mathbf{n}'' = 0$, then $t^{\mathbf{n}}(\mathbf{I}^{\mathbf{n}})_A\mathbf{D} = t^{\mathbf{n}'}\mathbf{I}'\mathbf{D}$ (by (b)) $= t^{\mathbf{n}'}\mathbf{I}'\mathbf{R}$ (by (a) and (c)). Hence, since $t^{\mathbf{n}'}\mathbf{I}'\mathbf{R}$ is a regular ideal, it follows that \mathbf{D} is a finite \mathbf{R} -module.
Q.E.D.

Proof of (4.1). Let J_1, \dots, J_h be regular ideals of R that generate Δ (so $\Delta = \{\mathbf{J}^{\mathbf{n}}; \mathbf{n} \in \mathbf{N}_h\}$, where $\mathbf{J} = (J_1, \dots, J_h)$), let $\mathbf{T} = \mathbf{R}(R, J_1, \dots, J_h, \phi_1, \dots, \phi_f)$, and let $\mathbf{E} = R[u_1, \dots, u_{f+h}, \{t^{\mathbf{n}}(\mathbf{J}^{\mathbf{n}}\Phi(\mathbf{n}''))_A; \mathbf{n} \in \mathbf{N}_{f+h}, \mathbf{n}' \in \mathbf{N}_h \text{ is such that } \mathbf{n}'(i) = \mathbf{n}(i) \text{ for } i = 1, \dots, h, \text{ and } \mathbf{n}'' \in \mathbf{N}_f \text{ is such that } \mathbf{n}''(i) = \mathbf{n}(h+i) \text{ for } i = 1, \dots, f\}]\}$. Then \mathbf{E} is a finite \mathbf{T} -module, by (4.4). Therefore, since $\phi_x \leq \phi_A$ for all filtrations ϕ (by hypothesis), $\mathbf{D} = R[u_1, \dots, u_{f+h}, \{t^{\mathbf{n}}(\mathbf{J}^{\mathbf{n}}\Phi(\mathbf{n}''))_x; \mathbf{n} \in \mathbf{N}_{f+h}, \mathbf{n}' \in \mathbf{N}_h \text{ is such that } \mathbf{n}'(i) = \mathbf{n}(i) \text{ for } i = 1, \dots, h, \text{ and } \mathbf{n}'' \in \mathbf{N}_f \text{ is such that } \mathbf{n}''(i) = \mathbf{n}(h+i) \text{ for } i = 1, \dots, f\}]\} \subseteq \mathbf{E}$, so it follows that \mathbf{D} is a finite \mathbf{T} -module. Therefore by (4.2.1) \Rightarrow (4.2.5) (where Φ (resp. Γ) of (4.2) (esp. (4.2.5)) is $(\phi_1, \dots, \phi_f, \{J_1^n\}_{n \geq 0}, \dots, \{J_h^n\}_{n \geq 0})$ (resp. (ϕ_1, \dots, ϕ_f))), it follows that $\mathbf{B} = R[u_1, \dots, u_f, \{t^{\mathbf{n}}(\Phi(\mathbf{n}))_x; \mathbf{n} \in \mathbf{N}_f\}]$ is a finite \mathbf{R} -module, where $\mathbf{R} = \mathbf{R}(R, \phi_1, \dots, \phi_f)$, so (4.2.5) \Rightarrow (4.2.6) shows that (4.1.1) holds.

Since \mathbf{R} is Noetherian and \mathbf{B} is a finite \mathbf{R} -module (where \mathbf{R} and \mathbf{B} are as in the preceding paragraph), the proofs of (4.1.2)–(4.1.4) are essentially the same as the proofs of (3.1.2)–(3.1.4).
Q.E.D.

(4.5) COROLLARY. *Let Δ be a finitely generated multiplicatively closed set of regular ideals of a Noetherian ring R and let $\Phi = (\phi_1, \dots, \phi_f)$ be a finite collection of Noetherian filtrations on R . Then there exists $K \in \Delta$ such that $(\Phi(\mathbf{n}))_A = \Phi(\mathbf{n})K : K$ for all $\mathbf{n} \in \mathbf{N}_f$.*

Proof. Let \mathbf{R} and \mathbf{B} be as in the proof of (4.1) (but with $\phi_x = \phi_A$). Then that proof shows that \mathbf{B} is a finite \mathbf{R} -module, so (4.2.1) holds, and hence the conclusion follows immediately from (4.3.2). Q.E.D.

(4.6) COROLLARY. *Let Δ be a finitely generated multiplicatively closed set of regular ideals of a Noetherian ring R and let $\mathbf{I} = (I_1, \dots, I_g)$ be a finite collection of ideals of R . Then for all large $\mathbf{k} \in \mathbf{P}_g$ it holds that $(\mathbf{I}^{\mathbf{n}+\mathbf{k}})_A = \mathbf{I}^{\mathbf{n}}(\mathbf{I}^{\mathbf{k}})_A$ for all $\mathbf{n} \in \mathbf{N}_g$, and if $\Phi = (\phi_1, \dots, \phi_f)$ is a finite collection of Noetherian filtrations on R , then there exists $K \in \Delta$ such that $(\mathbf{I}^{\mathbf{n}}\Phi(\mathbf{m}))_A = \mathbf{I}^{\mathbf{n}}\Phi(\mathbf{m})K : K$ for all $\mathbf{n} \in \mathbf{N}_g$ and $\mathbf{m} \in \mathbf{N}_f$.*

Proof. This follows immediately from (4.1.4) and (4.5). Q.E.D.

The final corollary in this section is an analog of (3.5) that holds for all Noetherian rings.

(4.7) COROLLARY. *Let R and $\phi \rightarrow \phi_x$ be as in (4.1) and let $\Phi = (\phi_1, \dots, \phi_g)$ be a finite collection of Noetherian filtrations on R . Then there exists $\mathbf{h} \in \mathbf{P}_g$ such that $(\Phi((\mathbf{n}+\mathbf{k})\mathbf{h}+\mathbf{r}))_x = (\Phi(\mathbf{h}))^{\mathbf{n}}(\Phi(\mathbf{k}\mathbf{h}+\mathbf{r}))_x$ for all large $\mathbf{k} \in \mathbf{P}_g$, for all $\mathbf{n} \in \mathbf{N}_g$, and for all $\mathbf{r} \in \mathbf{N}_g$ such that $\mathbf{h}(i) \leq \mathbf{r}(i) < 2\mathbf{h}(i)$ for $i = 1, \dots, g$.*

Proof. The proof is similar to the proof of (3.5), but use (4.1.3) in place of (3.1.3). Q.E.D.

5. A CHARACTERIZATION OF ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS

As noted at the start of Section 3, Sakuma and Okuyama showed in [9] that if I is an ideal of an analytically unramified semi-local ring R , then

$$(\mathbf{I}^{n+k})_a = \mathbf{I}^n(\mathbf{I}^k)_a \quad \text{for all large } k \text{ and for all } n \geq 0. \quad (\star)$$

In fact, they showed therein that (\star) holds for all regular ideals I of R if and only if R is analytically unramified. And if R is analytically unramified, then (3.1.3) and (3.1.4) generalize (\star) , so it follows that each of these conditions (for the case when $\phi_x = \phi_w$ (in (3.1.3)) or $I_x = I_a$ (in (3.1.4)))

also characterizes when R is analytically unramified. In this short section we use some of the other results of Sections 3 and 4 to give two new characterizations of when R is analytically unramified.

For (5.1), a filtration ϕ is said to be *regular* in case $\phi(1)$ is regular (equivalently, $\phi(n)$ is regular for all $n \geq 1$).

(5.1) THEOREM. *If R is a semi-local ring, then the following are equivalent:*

(5.1.1) *R is analytically unramified.*

(5.1.2) *For each multiplicatively closed set Δ of regular ideals of R and for each finite collection $\Phi = (\phi_1, \dots, \phi_g)$ of regular Noetherian filtrations on R there exists $K \in \Delta$ such that $(\Phi(\mathbf{n}))_\Delta = \Phi(\mathbf{n})K : K$ for all $\mathbf{n} \in \mathbf{N}_g$.*

(5.1.3) *For each finite collection $\Phi = (\phi_1, \dots, \phi_g)$ of regular Noetherian filtrations on R there exists a regular ideal K of R such that $(\Phi(\mathbf{n}))_\Delta = \Phi(\mathbf{n})K : K$ for all $\mathbf{n} \in \mathbf{N}_g$.*

Proof. Assume that (5.1.1) holds. Then since the ideals of Δ are regular, the proof of (3.4) shows that $\phi_\Delta \leq \phi_w$ for all filtrations ϕ on R . Therefore the second paragraph of the proof of (3.1) shows that $R[u_1, \dots, u_g, \{\mathbf{t}^n(\Phi(\mathbf{n}))_\Delta; \mathbf{n} \in \mathbf{N}_g\}]$ is a finite $\mathbf{R}(R, \phi_1, \dots, \phi_g)$ -module, so (4.2.1) holds, so the formula in (4.3.2) holds (which is the formula in (5.1.2)), and hence (5.1.1) \Rightarrow (5.1.2).

Assume that (5.1.2) holds and let I be a regular ideal of R . Now I is a reduction of I_Δ , so $(I_\Delta)^{n+1} = I(I_\Delta)^n$ for all large n . Therefore $I_\Delta \subseteq I(I_\Delta)^n : (I_\Delta)^n$, and it follows from the Cancellation Law (see [5, Sect. 6]) that $I \subseteq IJ : J \subseteq I_\Delta$ for all regular ideals J of R (since $(IJ : J)J = IJ$), and hence $I_\Delta = I(I_\Delta)^n : (I_\Delta)^n$. Therefore, since $(I_\Delta)^n$ is regular, it follows that if Δ is the set of all regular ideals of R , then $I_\Delta = I_\Delta$ for all regular ideals I of R , so $\phi_\Delta = \phi_a$ for all regular filtrations ϕ on R , and hence (5.1.2) \Rightarrow (5.1.3).

Finally, assume that (5.1.3) holds, let I be a regular ideal of R , and let $\phi = \{I^n\}_{n \geq 0}$. Then (5.1.3) (applied to ϕ in place of Φ) implies that $(I^n)_\Delta = I^n K : K$ for some regular ideal K of R , so since (4.3.2) \Rightarrow (4.3.1a), it follows that $(I^{n+k})_\Delta = I^n (I^k)_\Delta$ for all large k and for all $n \geq 0$. Therefore (\star) of the introduction of this section holds, so as noted in this introduction, [9] shows that (5.1.1) holds. Q.E.D.

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